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Poles of matrix elements of the resolvent connected with the range in an oscillator-boson coupling

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Abstract. In a simple model in which a harmonic oscillator is coupled to massless scalar bosons, we exhibit poles of matrix elements of the resolvent of the Hamiltonian which differ from those usually attributed to the excited levels of the isolated oscillator. We give reasons why one should expect such poles in atom-radiation interaction and explain how their importance depends on the range of the interaction.

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1 Introduction

The representation of atomic excited states has often been discussed. It is quite generally accepted that they appear in the theory as poles of matrix elements of the resolvent of the Hamiltonian and that to each state corresponds one pole. In fact, this view is built on a series of approximations.

The first one consists in considering emission or absorption profiles as infinitely narrow. Quantified excited states are thus introduced.

The second one consists in a perturbative treatment. Keeping the notion of excited state just defined, and giving it a more or less appropriate mathematical equivalent, this treatment allows Quantum Electrodynamics to yield characteristics of these states, such as their energies or their life-times. Calculations are limited to first orders in certain series in the coupling constant. They exhibit poles in the complex plane of the spectral parameter, which are associated to the energy levels of the atom. One often leans upon these results to conserve a meaning to the notion of excited state as extracted from the first approximation.

Beyond these calculations, rigorous results have recently been obtained in the non-relativistic case (see [1–3]). By examining the analytic structure of matrix elements of the resolvent, the authors are able to construct complex numbers associated to atomic excited states in a non-perturbative way. However that structure is only studied in neighbourhoods of the unperturbed energies of excited states and the analyticity only proved outside a cuspidal domain with its vertex at the complex point just mentioned. Some singularities might exist in regions not covered by this examination.

Although the cited work might strengthen the conviction that excited states do exist, at least mathematically, the situation does not appear completely clear to us, mainly for the reason that atomic levels are not straightforwardly derived from spectral lines, if their finite width is taken into account [4]. We rather believe that their existence, and their treatment by Quantum Mechanics or even in QED, might appear as a kind of approximation of order zero, fairly accurate for negligible width but unsatisfactory otherwise. Such an approximation might also obscure certain theoretical aspects. For these reasons we feel that further study of the poles of matrix elements of the resolvent is of interest.

The simple model of a harmonic oscillator coupled with massless scalar bosons allows one to take a step in that direction.

The Hamiltonian is

$$H(\lambda,\mu,g) = a^*a \otimes 1 + 1 \otimes \mu H_{\rm rad} + \lambda(a^* \otimes c(g) + a \otimes c^*(g)). \qquad (1)$$

We suppose $\|g\|_2 = 1$. We will obtain information on poles of matrix elements of the resolvent of $H(\lambda, 1, g)$. The parameter μ has been introduced because the spectrum of $H(\lambda, 0, g)$ is easy to derive, some properties of the resolvent of $H(\lambda,\mu,g)$ for small μ being then obtained by continuity [5,6]. This will give us a starting point for deriving the analytic structure of the matrix elements for greater values of μ.

More precisely, let us denote the fundamental and excited states of the oscillator by $|0\rangle$, $|1\rangle, \ldots |n\rangle$, as well as the corresponding states of the oscillator-field system when the field is in the vacuum state $|\Omega_{\rm rad}\rangle$, since there will be no ambiguity in the context. It is easy to see that $\langle 1|[H(\lambda,\mu,g) - z]^{-1}|1\rangle$ has a pole near $-d(\lambda) \sim -\lambda^2$ for λ small and μ small (see Proposition 1). It is not the one associated with the first excited state, which is near 1. In preceding studies [5,6], we also exhibited a pole of $\langle 2|[H(\lambda,\mu,g) - z]^{-1}|2\rangle$ near 1, in the second sheet,

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for λ small and μ smaller than a certain value, depending on λ. The important point is of course that this pole is not the pole of $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$. More generally, matrix elements $\langle n|[H(\lambda,\mu,g)-z]^{-1}|n\rangle$ would give other poles in that region, poles that still differ from the one associated with the excited level of energy 1. The problem we are interested in is: what becomes of these poles when μ increases up to 1?

Besides the fact that the mathematical problem may have an interest in itself, one may find a motivation for seeking an answer to the above-mentioned question when we transpose it in the atom-radiation case, replacing excited states of the oscillator by excited states of the atom. Do poles that are likely to be present when μ is small survive when μ reaches the value 1? Let us note that if they do not disappear, a conjecture which our study makes plausible, they are usually ignored. This can be justified by the following argument. Indeed, what matters is the magnitude of their imaginary part as compared to the width of the resonances. We will see that one can expect this ratio to be large in Electrodynamics. But these poles might become important at scales where the linewidths are large.

To elucidate this question, we will consider $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$ and study the motion of the poles in z when μ increases from 0 to 1, and particularly that of the one which is close to $-d(\lambda)$ for μ close to 0. Among the poles of all the above mentioned matrix elements, this one is particularly important since, according to a remark by Arai, the Hamiltonian of the model can be deduced directly from its restriction to the subspace generated by the vectors $|1\rangle\otimes|\Omega_{\rm rad}\rangle$ and $|0\rangle\otimes|\varphi\rangle$, $\varphi\in\mathcal{D}(H_{\rm rad})$ [7]. We will follow the displacement of this pole. The result of our study, Proposition 2, will be to give a function g for which this pole of $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$ does not disappear for $\mu=1$. We will relate it to a pole of that particular g. Proposition 1 recalls the situation for μ small. It has to be noted that the existence for μ small of such a pole of $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$ has nothing to do with the singularities of g, at least at first sight.

2 The result. A sketch of the proof and a numerical example

The poles of $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$ are the zeros of $f(\lambda,\mu,.)$, defined for $z\in\mathbf{C}\backslash\mathbb{R}^+$ by

$$f(\lambda,\mu,z) := 1-z-\lambda^2\int_{-\infty}^{\infty}\frac{g^2(p)}{\mu|p|-z}\mathrm{d}p. \qquad (2)$$

For $\mu\neq 0$, $f(\lambda,\mu,.)$ is multi-valued. Assuming that g is meromorphic, real, with possibly a finite number of non-real poles p_i, we can get the value at a point $z\neq\mu|p_i|$, $z\neq 0$, after crossing the cut once counterclockwise, by a contour deformation and this value will be denoted by $f_+(\lambda,\mu,z)$. We will assume that

$$C_1 := \int_{-\infty}^{\infty}\frac{g^2(p)}{|p|}\mathrm{d}p<\infty$$

and

$$C_2 := \int_{-\infty}^{\infty}\frac{g^2(p)}{|p|^2}\mathrm{d}p<\infty.$$

The starting point of our analysis is the following proposition which recalls well-known properties of f, especially its zeros for small μ. The important point is that whereas one zero is close to 1 if λ is small, there exists another one, close to 0. More precisely,

Proposition 1

$f(\lambda,0,.)$ is defined except at the pole $z=0$. On $]-\infty,0[$, it vanishes at $z_0(\lambda,0):=-d(\lambda)$, where $d(\lambda)=2^{-1}(\sqrt{1+4\lambda^2}-1)$. If $0<\mu\leq C_1\lambda^2$, then, on $]-\infty,0]$, $f(\lambda,\mu,.)$ vanishes at one point denoted by $z_0(\lambda,\mu)$. With the above value for $\mu=0$, $z_0(\lambda,.)$ is a continuous and increasing function on $[0,C_1\lambda^2]$ which vanishes for $\mu=C_1\lambda^2$. If $\mu>C_1\lambda^2$ and $x\leq 0$, $f(\lambda,\mu,x)\neq 0$ and $f_+(\lambda,\mu,x)\neq 0$.

Our main task is to follow this zero of $f(\lambda,\mu,.)$ as μ increases beyond the value $\mu_{\rm c}(\lambda)$. A difficulty is due to the fact that, for $\mu=\mu_{\rm c}(\lambda)$, the function $z_0(\lambda,.)$ reaches a branch point of $f(\lambda,\mu,.)$. This forces us to use analyticity properties of $f(\lambda,\mu,.)$ with caution. Due to this fact, we will be able to give an answer only in a case where we have an explicit expression for f. This will be possible with the function g below which allows the integration in (2) to be done explicitly. In the whole section we set

$$g(p)=\sqrt{2/\pi}\frac{p}{1+p^2}\,. \qquad (3)$$

In spite of its particularity, this choice has its interest from the physical point of view. Indeed, in Quantum Electrodynamics, functions which play the role of g are rational. Their poles may be given a physical meaning (see Sect. 3.4.1). With this choice of g, it will be interesting to look at the role played in our problem by poles of such functions.

For this particular g, our result is the following. We are able to follow the paths of four zeros of the multi-valued (if $\mu\neq 0$) function $f(\lambda,\mu,.)$ as μ varies. The two important ones for us are denoted by $z_0(\lambda,\mu)$ and $z_1(\lambda,\mu)$. Two others are denoted by $z_{0,+}(\lambda,\mu)$ and $z_0'(\lambda,\mu)$.

- The first one is the complex number associated to the first excited state of the harmonic oscillator. We denote it by $z_1(\lambda,\mu)$; it varies continuously from its value $z_1(\lambda,0)=1+d(\lambda)$. Its path when μ increases from 0 to 1 lies entirely in the second sheet, that is to say $z_1(\lambda,\mu)$ is in fact a zero of $f_+(\lambda,\mu,.)$. There is nothing new for that zero.
- The second one, which we denote by $z_0(\lambda,\mu)$, may be considered as associated to a kind of bound state, the fundamental state of the oscillator accompanied by one photon. It is the zero we focus on in the paper. We already mentioned it for $\mu\leq\mu_{\rm c}(\lambda)$ in Proposition 1, and it can be traced from its value $-d(\lambda)$ for $\mu=0$ to its value for $\mu=1$, across the value 0 for $\mu=\mu_{\rm c}(\lambda)=(2/\pi)\lambda^2$. This path has to be described with caution

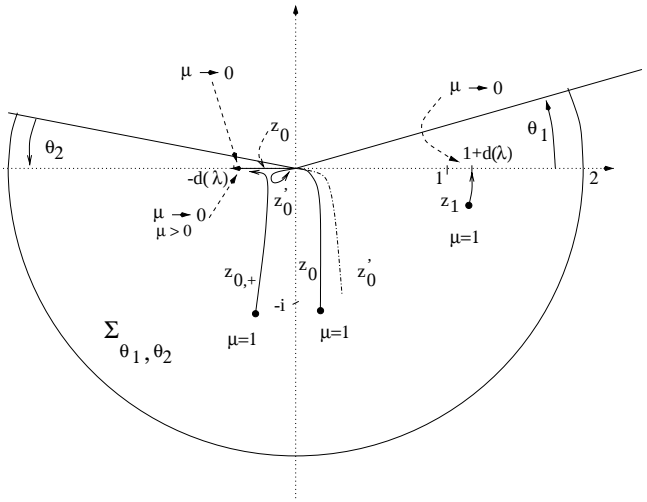

Fig. 1. Paths of zeros of $f(\lambda,\mu,.)$ (multi-valued if $\mu\neq 0$) when μ varies from 1 to 0.

since it crosses the branch point of f. $z_0(\lambda,\mu)$, starting from the value $z_0(\lambda,0)=-d(\lambda)$, stays in the principal sheet, that is to say is a (real) zero of f, until μ reaches the value $\mu_c(\lambda)$ where it comes to the branch point 0; then it enters the second sheet where it becomes a zero of $f_+(\lambda,\mu,z)$, the continuation of $f(\lambda,\mu,.)$ across the cut, which has to be considered only if $\lambda\neq 0$ and $\mu\neq 0$. This zero is linked to the pole of g, because the pole of g at $p=-\mathrm{i}$ produces a pole of the integral in (2) in the second sheet at $z=-\mathrm{i}\mu$, and, as a consequence, leads to f_+ having a zero distinct from the one which is close to 1, when λ is small (this zero may be expected to be close to the pole of the integral, when λ is small).

- The path of the third one, $z_{0,+}(\lambda,\mu)$, is also linked to the pole of g, although its existence for μ small does not depend on this pole (this is also true for $z_0(\lambda,\mu)$). This zero is analogous to the preceding one, but is a zero of f_+ instead of f. For small μ, as the two determinations f and f_+ at the same point z do not differ much, the zeros $z_0(\lambda,\mu)$ and $z_{0,+}(\lambda,\mu)$ of these two branches are neighbouring; they move apart when μ increases. $z_{0,+}(\lambda,\mu)$ tends to $-d(\lambda)$ when μ goes to 0 and does not have any particular value for $\mu=\mu_c(\lambda)$. $z_0(\lambda,\mu)$ and $z_{0,+}(\lambda,\mu)$ are close to the (double) pole of the integral, when λ is small.
- A fourth one is denoted by $z_0'(\lambda,\mu)$. It does not exist for $\mu=0$, but tends to 0 when μ goes to 0, and also when μ goes to $\mu_c(\lambda)$.

The paths described by these zeros when μ varies from 0 to 1 are shown in Figure 1.

More precisely, our result may be stated in the following terms.

Proposition 2

Let $g(p)=\sqrt{2/\pi}p/(1+p^2)$. Denote $\Sigma_-:=\{z;\ |z|\leq 2,\ \Im z\leq 0\}$.

(1) If $\lambda=0$, $f(\lambda,\mu,.)$ has one and only one zero; it is at $z=1$.

(2) There exists $\lambda_{\max}>0$ such that, if $0<\lambda<\lambda_{\max}$, then,
   (a) for $\mu=0$, $f(\lambda,\mu,.)$ is mono-valued and has two zeros: $z_0(\lambda,0)=-d(\lambda)$ and $z_1(\lambda,0)=1+d(\lambda)$;
   (b) for $0<\mu<\mu_c(\lambda)=(2/\pi)\lambda^2$, $f(\lambda,\mu,.)$ has one zero $z_0(\lambda,\mu)$ on $\mathbb{R}^-$ and, in Σ_-, $f_+(\lambda,\mu,.)$ has three and only three zeros, $z_1(\lambda,\mu)$, $z_{0,+}(\lambda,\mu)$ and $z_0'(\lambda,\mu)$, satisfying
$$\lim_{\mu\to 0^+} z_1(\lambda,\mu)=1+d(\lambda),$$
$$\lim_{\mu\to 0^+} z_{0,+}(\lambda,\mu)=-d(\lambda),$$
$$\lim_{\mu\to 0^+} z_0'(\lambda,\mu)=0,\ \lim_{\mu\to\mu_c(\lambda)} z_0'(\lambda,\mu)=0;$$
   (c) for $\mu_c(\lambda)\leq\mu\leq 3/2$ and $z\in\Sigma_-$, $f_+(\lambda,\mu,.)$ has three zeros, $z_1(\lambda,\mu)$, $z_0(\lambda,\mu)$ and $z_{0,+}(\lambda,\mu)$, which are continuous functions on $[\mu_c(\lambda),3/2]$. $z_0(\lambda,\mu_c(\lambda))=0$; thus $z_0(\lambda,\mu)$ connects to the zero of f described in Proposition 1 when $0\leq\mu\leq\mu_c(\lambda)$. $z_{0,+}(\lambda,\mu_c(\lambda))\neq 0$. Besides, for fixed non-zero μ, $\lim_{\lambda\to 0} z_1(\lambda,\mu)=1$, $\lim_{\lambda\to 0} z_0(\lambda,\mu)=\lim_{\lambda\to 0} z_{0,+}(\lambda,\mu)=-\mathrm{i}\mu$.

2.1 A sketch of the proof

We only give here a sketch of the proof in order not to enter too much into technicalities. Some are unavoidable. They are due to the fact that the branch point at $z=0$ prevents us from using analyticity of $f(\lambda,\mu,.)$ (and continuity with respect to μ) to follow the zeros by continuity, according to Hurwitz theorem. Indeed, some of them go to zero when μ varies in a neighbourhood of $\mu_c(\lambda)$ or of 0. It is thus necessary to introduce small (closed) disks D_r around $z=0$, setting $\Sigma_-^r:=\Sigma_-\setminus\overset{\circ}{D}_r$.

The study of zeros of f_+ is replaced by the study of zeros of $N_+(\lambda,\mu,.)$, the analytic continuation across the positive real axis of $N(\lambda,\mu,.)$, which is defined, for $\mu=0$, by

$$N(\lambda,0,z):=z(z(1-z)+\lambda^2) \tag{4}$$

and, for $\mu\neq 0$, by

$$N(\lambda,\mu,z):=(z+\mathrm{i}\mu)^2 f(\lambda,\mu,z). \tag{5}$$

We already mentioned that, for $\mu\neq 0$, $f_+(\lambda,\mu,.)$ has a (double) pole at $z=-\mathrm{i}\mu$, a consequence of the existence of a pole of g at $p=-\mathrm{i}$. This is why the multiplication of f by $(z+\mathrm{i}\mu)^2$ does not introduce spurious zeros of f_+, except for $\mu=0$. (Of course it introduces zeros of N at $z=-\mathrm{i}\mu$, which are not zeros of f. We are not concerned with those. However, when λ is small, the difference between N_+ and N being then small, the zeros of N_+ are close to these zeros $z=-\mathrm{i}\mu$ of N.) One has

$$N(\lambda,\mu,z):=\mu\Big(\frac{2\lambda^2}{\pi}-\mu\Big)+z\Big(z(1-z)+\lambda^2 + \mu(\mu+2\mathrm{i}(1-z))+\frac{4\lambda^2}{\pi}b(\mu,z)\Big), \tag{6}$$

with $b(\mu, z)$ defined in $\mathbf{C}\backslash\mathbb{R}^+$ by $b(\mu, \mathrm{i}\mu) := -\mathrm{i}/2$ and

$$b(\mu,z) = -\mu\frac{1}{z-\mathrm{i}\mu} + \mu\frac{z}{(z-\mathrm{i}\mu)^2}\log z - \frac{z}{(z-\mathrm{i}\mu)^2}\mu\log(\mathrm{i}\mu), \quad \text{for } z \neq \mathrm{i}\mu. \tag{7}$$

In this formula, the branch point of f, and thus of N, at $z = 0$ appears in the logarithm, taken with $\log z = \log|z| + \mathrm{i}\arg z$, with $0 < \arg z < 2\pi$.

The study of zeros of $N_+(\lambda, \mu, .)$ is carried with the help of two tools. One is the formula

$$n_{\gamma_r} := (2\mathrm{i}\pi)^{-1}\int_{\gamma_r}\frac{\partial_z N_+(\lambda,\mu,z)}{N_+(\lambda,\mu,z)}\,\mathrm{d}z, \tag{8}$$

which gives the number of zeros of N_+ inside the boundary γ_r of Σ_-^r, provided $N_+(\lambda, \mu, .)$ does not vanish on γ_r. The other one is Rouche's theorem. N_+ being written as the sum of a main part $N_+ - \Delta$ and a correction Δ, this theorem relates the number of zeros of N_+ to the number of zeros of $N_+ - \Delta$, inside a contour on which $|\Delta| < |N_+ - \Delta|$ is true. Several decompositions will be used, in which the zeros of $N_+ - \Delta$ will be more easily controlled that those of N_+.

We distinguish three cases, according to whether $\mu > \mu_\mathrm{c}(\lambda)$, $\mu = \mu_\mathrm{c}(\lambda)$, or $\mu < \mu_\mathrm{c}(\lambda)$.

(i) $\mu > \mu_\mathrm{c}(\lambda)$. Due to the non-zero distance between μ and $\mu_\mathrm{c}(\lambda)$, we can show that there exists a value $\lambda_{\max}$ of λ such that, for (λ, μ) in $\mathcal{R} := \{(\lambda, \mu);\ 0 \leq \lambda \leq \lambda_{\max}, \mu_\mathrm{c}(\lambda) < \mu \leq 3/2\}$, then $N_+(\lambda, \mu, .)$ does not vanish on γ; moreover there exists $\epsilon(\lambda, \mu) > 0$ such that for $(\lambda, \mu) \in \mathcal{R}$, $N_+(\lambda, \mu, .)$ does not vanish in $\Sigma_- \cap D_{\epsilon(\lambda,\mu)}$, and consequently does not vanish on $\gamma_{\epsilon(\lambda,\mu)}$. Then we can define

$$n_{\gamma,\epsilon(\lambda,\mu)}(\lambda,\mu) := (2\mathrm{i}\pi)^{-1}\int_{\gamma_{\epsilon(\lambda,\mu)}}\frac{\partial_z N_+(\lambda,\mu,z)}{N_+(\lambda,\mu,z)}\,\mathrm{d}z. \tag{9}$$

It is the number of zeros of N_+ inside $\gamma_{\epsilon(\lambda,\mu)}$, which is also the number of zeros of N_+ inside γ. We simply denote this quantity by $n(\lambda, \mu)$. It can then be shown that $n(., .)$ is continuous at each point (λ_0, μ_0) of $\mathcal{R}$. And thus it is a constant. Since $n_\gamma(0, \mu) = 3$, the number of zeros of N_+ inside Σ_- is three.

Let us denote these three zeros by $z_0(\lambda, \mu)$, $z_{0,+}(\lambda, \mu)$ and $z_1(\lambda, \mu)$. From Hurwitz theorem, they are continuous functions of (λ, μ) in $\mathcal{R}$. For non-zero fixed μ, we shall denote the zero satisfying $\lim_{\lambda\to 0} z_1(\lambda, \mu) = 1$ as $z_1(\lambda, \mu)$, and $z_0(\lambda, \mu)$ and $z_{0,+}(\lambda, \mu)$ will denote the two other zeros, satisfying $\lim_{\lambda\to 0} z_0(\lambda, \mu) = \lim_{\lambda\to 0} z_{0,+}(\lambda, \mu) = -\mathrm{i}\mu$. The affectation of the notations $z_0(\lambda, \mu)$ and $z_{0,+}(\lambda, \mu)$ to the two zeros will be made precise later on.

(ii) $\mu = \mu_\mathrm{c}(\lambda)$. It is clear from (6) that $z = 0$ is a zero of $N_+(\lambda, \mu_\mathrm{c}(\lambda), z)$. The others are those of the function

$$h_+(\lambda,z) := z(1-z) + \lambda^2 + \mu_\mathrm{c}(\lambda)(\mu_\mathrm{c}(\lambda) + 2\mathrm{i}(1-z)) + \frac{4\lambda^2}{\pi}b_+(\mu_\mathrm{c}(\lambda), z), \tag{10}$$

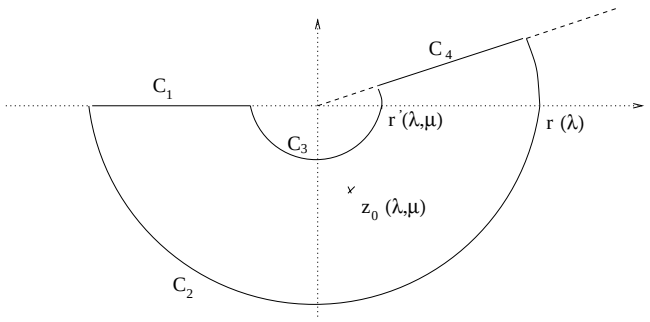

Fig. 2. The contour C used in the application of Rouche's theorem.

which is continuous at $z = 0$, in the sector Σ_-. (The index $+$ in functions indicates one crossing of the positive axis.) We are going to prove the following statement (A): if μ, greater than $\mu_\mathrm{c}(\lambda)^+$, is in a sufficiently small neighbourhood $\mathcal{V}$ of $\mu_\mathrm{c}(\lambda)$, then one and at most one zero of N_+ inside Σ_- lies in a certain disk $\overset{\circ}{D}_{r(\lambda)}$, the radius of which does not depend on μ in $\mathcal{V}$. It is this zero which has been denoted by $z_0(\lambda, \mu)$. This will imply that the two other zeros we found before are outside $\overset{\circ}{D}_{r(\lambda)}$, and, therefore will stay in that region when $\mu \to \mu_\mathrm{c}(\lambda)^+$. We will thus get the three zeros of N_+ inside Σ_-, in the case $\mu = \mu_\mathrm{c}(\lambda)$: two outside $D_{r(\lambda)}$ and one at 0. To prove (A), we use Rouche's theorem, noting that, for $z \neq \mathrm{i}\mu$, N_+ may be written as the sum $Q(\lambda, \mu, z) + \Delta(\lambda, \mu, z)$ of a rational function

$$Q(\lambda,\mu,z) := \mu(\mu_\mathrm{c}(\lambda) - \mu) + zQ_1(\lambda,\mu,z), \tag{11}$$

where

$$Q_1(\lambda,\mu,z) := z(1-z) + \lambda^2 + \mu^2 + 2\mathrm{i}\mu(1-z) - \frac{4\lambda^2}{\pi}\frac{\mu}{z-\mathrm{i}\mu} - \frac{4\lambda^2}{\pi}\mu\log(\mathrm{i}\mu)\frac{z}{(z-\mathrm{i}\mu)^2}, \tag{12}$$

and a multi-valued part

$$\Delta(\lambda,\mu,z) := \frac{4\lambda^2\mu}{\pi}\frac{z^2(\log z - 2\mathrm{i}\pi)}{(z-\mathrm{i}\mu)^2}\,. \tag{13}$$

This enables us to relate the zeros of N_+ to those of $Q(\lambda, \mu, z)$. The closed contour C in Rouche's theorem is constructed in such a way that it leaves the singularity $z = 0$ outside, $Q(\lambda, \mu, .)$ vanishes only once inside C and $|\Delta| < |Q|$ on C. C is made of four parts C_i which are drawn in Figure 2. (On this figure the contour does not lie in Σ_- since the part C_4 is not on the positive real axis but moved on a ray making a certain positive angle θ_1 with this axis. This complication, which forces us to introduce a new notation $\hat{f}$, the continuation of f_+ inside the contour, is due to the fact that we do not know how to prove $|\Delta| < |Q|$ on the reals. So, in the following, Σ_- should in fact be replaced by $\Sigma_{\theta_1,0}$ (see Fig. 1 for the notation) and N_+ by $\hat{N}$. But N does not vanish for $\Im z > 0$).

An other property that C has to satisfy in order to get (A) is that N_+ does not vanish inside $\Sigma_- \cap D_{r'(\lambda,\mu)}$. Since $z_0(\lambda,\mu)$ is expected to reach 0 when $\mu \to \mu_c(\lambda)^+$, the contour has a part C_3 which depends on μ and shrinks to 0 when $\mu \to \mu_c(\lambda)$. The proof of the inequality $|\Delta| < |Q|$ uses the behaviour of Δ for small z and supposes that the contour C containing the zero is contained in a small neighbourhood of 0, which is achieved by taking μ sufficiently close to $\mu_c(\lambda)$. Since it is a bit technical, this proof is not reproduced here. On this line, the number of zeros of N_+ (in fact $\hat{N}$) inside C can be proved to be one and therefore (A) is proved.

(iii) $\mu < \mu_c(\lambda)$. We again look at the number of zeros of N_+ by means of formula (9), but this time using continuity with respect to μ, in particular at $\mu = 0$. The fact that for $\mu = 0$, one of the zeros of N_+ is on the real negative axis and another one on the positive axis forces us to enlarge Σ_- to $\Sigma_{\theta_1,\theta_2}$. A complication comes from the fact that each end of the interval $[0, \mu_c(\lambda)[$ produces a zero at $z = 0$. This forces us to separate the study in three parts.

For μ in a neighbourhood of 0, formula (6) leads us to expect a zero of N_+ near $z = 0$. Let us sketch how one can prove its existence. By looking at the zeros of the approximating function

$$F_{\rm app}(\lambda,\mu,z) := \lambda^2 \left(\frac{-2\mu^3}{\pi} - 2z^2\mu \left(\mathrm{i} + \frac{1}{\pi} \right) + z(z - \mathrm{i}\mu)^2 + \frac{4\mu}{\pi} z^2 \left(log\frac{z}{\mu} - 2\mathrm{i}\pi \right) \right),$$

which is homogeneous in (μ, z), and applying Rouche's theorem, we can show that for any $\rho \in]0, \min\{1, \lambda^2/12\}]$, if μ is smaller than a certain $\mu_2(\rho)$, $N_+(\lambda,\mu,.)$ does not vanish on the boundary of $\Sigma_{\theta_1,\theta_2} \backslash D_\rho$; the integral in (9) thus makes sense on this contour. If $N_+(\lambda,\mu,.)$ is restricted to $\Sigma_{\theta_1',\theta_2'}$ for some sufficiently small θ_1', θ_2', it can also be shown that inside D_ρ, $N_+(\lambda,\mu,.)$ has exactly one zero, $z_0'(\lambda,\mu)$. Outside D_ρ, the number of zeros of $N_+(\lambda,\mu,.)$ is two (by continuity because it is 2 for $\mu = 0$). These two last zeros are denoted by $z_{0,+}(\lambda,\mu)$ and $z_1(\lambda,\mu)$. On the whole, we thus have three zeros: $z_{0,+}(\lambda,\mu)$, $z_1(\lambda,\mu)$ and $z_0'(\lambda,\mu)$ for μ close to zero. They are in fact in Σ_-.

For μ in a neighbourhood of $\mu_c(\lambda)$, *i.e.* for $\epsilon(\lambda,\mu) := \mu_c(\lambda) - \mu$ small and positive, we may also expect a zero of N close to 0, from (6). To prove its existence we lean upon the existence in this region of a zero of the following neighbouring function:

$$N_{\rm app}(\lambda,\mu,z) := (\mu_c(\lambda) - \mu)\mu + z\left(\lambda^2 + \mu^2 - 2\mathrm{i}(\mu_c(\lambda) - \mu)\right).$$

We have $N_+(\lambda,\mu,z) = N_{\rm app}(\lambda,\mu,z) + z^2\, R(\lambda,\mu,z)$, where

$$R(\lambda,\mu,z) = 1 - z - 2\mathrm{i}\mu + \frac{4\lambda^2}{\pi}\left(\frac{\mathrm{i}}{z - \mathrm{i}\mu} + \frac{\mu \log z - 2\mathrm{i}\mu\pi - \mu\log(\mathrm{i}\mu)}{(z - \mathrm{i}\mu)^2} \right).$$

For $\alpha > 0$, $z^\alpha R(\lambda,\mu,.)$ is bounded in $\Sigma_{\theta_1,\theta_2}$, uniformly for μ in a neighbourhood of $\mu_c(\lambda)$.

$N_{\rm app}(\lambda,\mu,.)$ has a unique zero located at

$$\hat{z}_0'(\lambda,\mu) := -\frac{\epsilon(\lambda,\mu)}{\lambda^2 + (\mu_c(\lambda) - \epsilon(\lambda,\mu))^2 - 2\mathrm{i}\epsilon(\lambda,\mu)}\mu.$$

The zero of $N_+(\lambda,\mu,.)$ is then deduced by Rouche's theorem in the following way. For $0 < \eta_1 < \eta_2$, we denote by $\mathcal{A}_{\eta_1,\eta_2}$ the closed piece of ring limited on the one hand by the two circles of radii respectively η_1 and η_2 and on the other hand by the semi-axis θ_1 and θ_2 of Figure 1. For $\mu \in [(1/2)\mu_c(\lambda), \mu_c(\lambda)[$, $\hat{z}_0'(\lambda,\mu)$ lies inside $\mathcal{A}_{\epsilon_1(\lambda,\mu),\epsilon_2(\lambda,\mu)}$, where

$$\epsilon_1(\lambda,\mu) := \frac{1}{\pi}\frac{1}{2 + 4\lambda^2/\pi^2}\epsilon(\lambda,\mu)$$
$$\text{and } \epsilon_2(\lambda,\mu) := (2/\pi)\epsilon(\lambda,\mu).$$

We can show that there exists η such that, for $\mu \in [\mu_c - \eta, \mu_c[$ (which implies $\eta > \epsilon_2(\lambda,\mu)$), and z on the boundary of $\mathcal{A}_{\epsilon_1(\lambda,\mu),\eta} \supset \mathcal{A}_{\epsilon_1(\lambda,\mu),\epsilon_2(\lambda,\mu)}$,

$$\left| \frac{z^2 R(\lambda,\mu,z)}{N_{\rm app}(\lambda,\mu,z)} \right| < 1.$$

So, by Rouche's theorem, $N_+(\lambda,\mu,z)$ has one and only one zero inside $\mathcal{A}_{\epsilon_1(\lambda,\mu),\eta}$. As

$$\left| \frac{z^2 R(\lambda,\mu,z)}{N_{\rm app}(\lambda,\mu,z)} \right| < 1$$

is also true for $|z| < \epsilon_1(\lambda,\mu)$, this zero is the only one in $\Sigma_- \cap D_\eta$. For the moment we call it $z_0''(\lambda,\mu)$, but we will see that it can be linked to $z_0'(\lambda,\mu)$ precedingly defined on $[0, \mu_2(\rho)]$; then it will simply be called $z_0'(\lambda,\mu)$. As we can also prove that

$$\left| \frac{z^2 R(\lambda,\mu,z)}{N_{\rm app}(\lambda,\mu,z)} \right| < 1$$

on the boundary of $\mathcal{A}_{\epsilon_1(\lambda,\mu),\epsilon_2(\lambda,\mu)}$, $z_0''(\lambda,\mu)$ lies in fact inside $\mathcal{A}_{\epsilon_1(\lambda,\mu),\epsilon_2(\lambda,\mu)}$ and thus goes to 0, when μ goes to $\mu_c(\lambda)$. At this stage, we got two germs of zeros located at $z = 0$, one for $\mu = 0$ and one for $\mu = \mu_c(\lambda)$. Two other zeros $z_{0,+}(\lambda,.)$ and $z_1(\lambda,.)$ arise by continuity from the non-zero values $z_{0,+}(\lambda,\mu_c(\lambda))$ and $z_1(\lambda,\mu_c(\lambda))$ obtained in (i). The fact that there are no other zeros in Σ_- than these three will be a consequence of what follows.

Let us now consider intermediate values of μ, more precisely $\mu \in [(1/2)\mu_2(\eta), \mu_0]$ with $\mu_0 \in [(1/2)\mu_2(\eta), \mu_c(\lambda)[$, η being defined above. (6) shows that there exists an $\epsilon(\mu_0)$ such that N_+ does not vanish inside a disc of radius $\epsilon(\mu_0)$. In $\Sigma_- \backslash D_{\epsilon(\mu_0)}$, a region on the boundary of which N_+ does not vanish, the number of zeros is a continuous function of μ, and thus a constant function. For $\mu \in [(1/2)\mu_2(\eta), \mu_2(\eta)]$, we saw that this constant value is 3. So it is still 3 all over the interval $[(1/2)\mu_2(\eta), \mu_0]$. Let us make their locations a bit more precise.

One and only one of the zeros has been seen to be inside D_η, for μ in $](1/2)\mu_2(\eta), \mu_2(\eta)]$ (we called it z_0') or in $[\mu_c(\lambda) - \eta, \mu_c(\lambda)]$ (we called it z_0''). The two intervals

Table 1. Approximate values of the zeros of N as μ varies.

μ	$10^3 z_0(\lambda,\mu)$	$10^3 z_{0,+}(\lambda,\mu)$	$10^3 z_0'(\lambda,\mu)$	$10^3(z_1(\lambda,\mu)-1)$
10^{-5}	-9.82	$-9.82-0.1$i	$-0.0018-0.0018$i	~ 9.90
10^{-4}	-9.36	$-9.44-0.8$i	$-0.018-0.017$i	$9.91-0.004$i
0.001	-6.8	$-11-6$i	$-0.17-0.16$i	$10-0.04$i
0.005	-1.1	$-17-15$i	$-0.41-0.26$i	.
0.006	-0.3	$-18-17$i	$-0.21-0.066$i	.
0.0063	-0.04	$-18-17$i	$-0.043-0.003$i	.
μ_c	0	$-18-17$i	0	.
0.007	$0.4-0.06$i	$-18-18$i	third sheet (?)	.
0.01	$1.8-0.9$i	$-21-22$i	.	$10-0.4$i
0.1	$24-69$i	$-43-128$i	.	$11-4$i
0.5	$75-443$i	$-85-544$i	.	$5-13$i
1	$110-949$i	$-110-1041$i	.	$-4-10$i

are *a priori* disjoint. Let us explain now how the two functions $z_0'(\lambda,.)$ and $z_0''(\lambda,.)$ can be joined continuously. We can show that there exists $r>0$ such that, for μ varying in $]0,\mu_c(\lambda)]$, the circle with radius r is not crossed by the paths of the zeros of $N_+(\lambda,\mu,z)$ in Σ_-; for small μ, $z_{0,+}(\lambda,\mu)$ is outside the circle and $z_0'(\lambda,\mu)$ is inside. Thus $z_0'(\lambda,.)$ and $z_0''(\lambda,.)$ can be patched into a unique function $z_0'(\lambda,.)$ which values stay inside the circle. Thus the three zeros of N_+, $z_{0,+}(\lambda,\mu)$, $z_1(\lambda,\mu)$ and $z_0'(\lambda,\mu)$, that we saw for μ small extend as functions of μ in $]0,\mu_c(\lambda)]$, being the only zeros of N_+ in Σ_-.

The continuity with respect to μ in $[0,1]$ of the three functions $z_0(\lambda,\mu)$, $z_{0,+}(\lambda,\mu)$ and $z_1(\lambda,\mu)$ is a consequence of Hurwitz theorem, except for the continuity of $z_0(\lambda,\mu)$ at $\mu_c(\lambda)$ which is seen directly. In the same way, it can be shown that $z_0'(\lambda,\mu)$ is continuous on $]0,\mu_c(\lambda)[$. This ends the sketch of the proof.

Let us now make a comment about what happens at $\mu=\mu_c(\lambda)$. We saw that the two parts $\mu<\mu_c(\lambda)$ and $\mu>\mu_c(\lambda)$ of $z_0(\lambda,.)$ meet at 0 for $\mu=\mu_c(\lambda)$. But $z_0'(\lambda,.)$ (which so far we only defined for $\mu<\mu_c(\lambda)$) also goes to 0 when μ goes to $\mu_c(\lambda)$. If μ describes half a circle in the upper half-plane near this critical value $\mu_c(\lambda)$, we can define extensions in the complex of the functions $z_0(\lambda,.)$ and $z_0'(\lambda,.)$. Although we have not completely proved it, we expect that the extension of $z_0(\lambda,.)$ connects two pieces in two parts of the function $z_0(\lambda,.)$ we defined for real μ, while the extension of $z_0'(\lambda,.)$ would branch on a zero of f_{++} for $\mu\geq\mu_c(\lambda)$. (f_{++} is the continuation of f in the lower half-plane after two crosses of the positive axis.) The study in the complex would thus legitimate our choice of what we called z_0.

The paths followed by the zeros are depicted qualitatively in Figure 1. In the next section, we illustrate the motion as μ varies of the zeros we found by taking a particular value for λ.

2.2 A numerical example

We nail down λ to $\lambda=10^{-1}$.

Then $\mu_c = 2/100\pi \sim 0.0063662$ and $d(\lambda) = 2^{-1}(\sqrt{1+4\lambda^2}-1)\sim 0.00990125$. Approximate values of the zeros obtained by computer are given in Table 1.

3 Conclusion, comments and perspectives

3.1 The result

By our example, we progressed towards an answer to the question asked at the end of the Introduction, by proving the following. Matrix elements of the resolvent of $H(\lambda,1,g)$ may have poles not usually associated to the excited states of the uncoupled oscillator. In Section 2, it was proved that one of these poles, $z_0(\lambda,1)$, could be traced continuously as a pole of a matrix element of the resolvent of $H(\lambda,\mu,g)$, μ varying from 0 to 1. For μ small, we try to give these poles a physical interpretation in Section 3.2.

Although this property has only been proved for a particular g, it may be expected more generally, in view of formulas (2), that a pole $p=p_0$ of g in the complex plane is to create zeros of f_+ and consequently poles of $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$ near $z=p_0$ (consider for instance the model $f(\lambda,z)=1-z-\lambda^2(z-p_0)^{-1}$, where a pole of f at $z=p_0$ creates a zero of f, close to p_0 if λ is small, and thus *a priori* distinct from 1). Thus poles of g are particularly important. Therefore we will suppose in the following that our result does not depend much on the form of the function g and extends to functions more general than $g(p)=\sqrt{2/\pi}p(1+p^2)^{-1}$.

With this assumption, for a class of rational g's, matrix elements of the resolvent of the Hamiltonian would exhibit two kinds of poles. The first kind would consist of the poles usually associated to the levels of the uncoupled oscillator. We focused on the first level. The second kind of poles would consist of poles related to poles of g,

in the same way as $z_0(\lambda,1)$ and $z_{0,+}(\lambda,1)$ are related to the pole $-\mathrm{i}$ in the example we treated. It could be asked whether $z_0(\lambda,1)$ and $z_{0,+}(\lambda,1)$ would not simply be accidents due to the non analyticity of g, with no physical meaning (incidentally, this view would suppose that poles of g do not have any physical meaning; we will discuss this point later on in Sect. 3.4.1). Against that possibility, let us remark that the existence of $z_0(\lambda,\mu)$ for $\mu \leq \mu_c(\lambda)$ is proved in Proposition 1 under quite general conditions on g. The fact that possible poles of g do not play any role in that proof seems to indicate that the pole $z_0(\lambda,\mu)$ is not "accidental".

3.2 An interpretation

For small μ, we already proposed an interpretation for the presence of that pole (see [6]). It was based on a continuity argument and on the spectrum of $H(\lambda,0,g)$. Indeed, for $\mu=0$, 0 is an infinitely degenerated eigenvalue of $a^*a\otimes 1$ which splits with the perturbation $\lambda(a^*\otimes c(g)+a\otimes c^*(g))$. The two eigenvalues 0 and $z_0(\lambda,0)$ are distinct points in the set generated by this splitting. The situation for μ small may be guessed by continuity. In other words, if we consider a 1-boson state of the oscillator-field system, $|0,\gamma\rangle$, and the state $|0,\Omega_{\mathrm{rad}}\rangle$ (the comma replaces a tensor product), we see that, if the energy of the boson is small, these two states have neighbouring energies and the interaction separates them, due to the coupling between $|0,\gamma\rangle$ and $|1,\Omega_{\mathrm{rad}}\rangle$ by absorption of one boson.

An other point of view may cast some light. The eigenvalues of (1) are the same as those of

$$H(\lambda,1,g_\mu) = a^*a\otimes 1 + 1\otimes H_{\mathrm{rad}} + \lambda(a^*\otimes c(g_\mu) + a\otimes c^*(g_\mu)), \qquad (14)$$

where $g_\mu(p) := \mu^{-1/2} g(\mu^{-1}p)$. Restricted to the space of states with at most one boson, the model is the one considered in [8], if the extra μ-dependence that we introduced is ignored. With that change of the Hamiltonian, the μ-dependence permits to vary the half-height width of the coupling function, since

$$\mathrm{width}(g_\mu) = \mu\,\mathrm{width}(g). \qquad (15)$$

The function which now plays the role of what we called $f_g(\lambda,\mu,z)$ (we make the dependence of (2) in g explicit) is $f_1(\lambda,\mu,z) := f_{g_\mu}(\lambda,1,z)$. Making μ small amounts to considering a coupling function λg_μ peaked near 0; for the coupling with the oscillator, the continuum may thus be said to be nearly reduced to a unique discrete state, and as a consequence, the evolution of the state $|1,\Omega_{\mathrm{rad}}\rangle$ close to a Rabi oscillation between 2 states. One of them is the eigenvector of (14) corresponding to $z_0(\lambda,\mu)$.

Our study allows to follow this eigenvalue which moves into the complex plane when μ increases, that is to say when the continuum is enlarged.

The same mechanism would operate for other eigenvalues. In reference [6], we were concerned with the eigenvalue 1. The 1-boson state $|1,\gamma\rangle$ and the 0-boson state $|1,\Omega_{\mathrm{rad}}\rangle$ have neighbouring energies if the energy of the boson is small. The interaction splits up the degeneracy and that would reflect in the presence of two distinct poles near 1, one being the usual pole of $\langle 1|[H(\lambda,\mu,g)-z]^{-1}|1\rangle$, the other being a pole of $\langle 2|[H(\lambda,\mu,g)-z]^{-1}|2\rangle$.

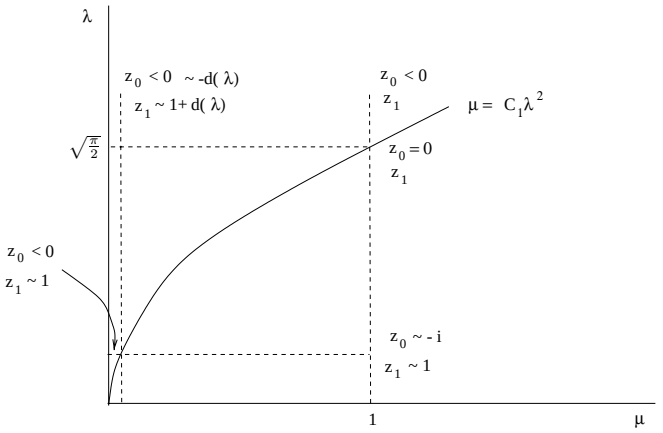

Fig. 3. Dependence with respect to (λ,μ) of the two main zeros of $f(\lambda,\mu,.)$.

3.3 Connection with critical and strong coupling

The expression (14) allows to connect our study to the discussion in [8] which concerns the effect of the coupling strength λ on the type of time evolution of probabilities amplitude in the Friedrichs model. In [8], one finds a graphic study of the zeros of the function $\Re f_1$, on the real axis. A property that is contained in Proposition 1 plays a role in that study: there exists a critical value $\lambda_c(\mu) = (\mu/C_1)^{1/2}$ of λ, for which $\Re f_1(\lambda,\mu,0)=0$, and such that, for $\lambda>\lambda_c(\mu)$, $\Re f_1(\lambda,\mu,.)$ has a negative zero, which is in fact a zero of $f_1(\lambda,\mu,.)$. The critical relation between λ and μ appeared in our study under the form $\mu=\mu_c(\lambda)$. The zero is interpreted in [8] in the following way. μ being fixed (equal to 1, in fact), as the coupling constant λ increases from 0 to a value slightly greater than $\lambda_c(\mu)$, the energy of the first excited state is lowered from 1 to a value which becomes negative after λ crosses $\lambda_c(\mu)$; it is the energy of a stable state. This interpretation should be reconsidered in the light of our result, which concerns zeros of f_1 instead of real zeros of $\Re f_1$.

Our study seems to indicate that the zero of f_1 which becomes real negative when λ increases from 0 to a value greater than $\lambda_c(\mu)$, μ being constant, is $z_0(\lambda,\mu)$, rather than $z_1(\lambda,\mu)$. $z_1(\lambda,\mu)$ moves in the second sheet. $z_0(\lambda,\mu)$ is not considered in [8] for small coupling, but probably manifests its effects in the position of the zeros of $\Re f_1$ observed for strong coupling. This point should be examined more thoroughly in a future study. In any case, the presence of one or three zeros of $\Re f_1$ on the real line is not incompatible with f_1 having only two zeros in the complex plane, as can be seen on the example $f(\lambda,\mu,z) = z-1+\lambda(z-\mathrm{i}\mu)^{-1}$. (Remember the third zero,

$z_{0,+}(\lambda,\mu)$, is not a zero of a third type since it is a zero of $f_{1,+}$, a continuation of f, connected with $z_0(\lambda,\mu)$). Figure 3 indicates roughly the position in the complex plane of the two zeros, according to different values of (λ,μ).

3.4 A possible generalisation

3.4.1 The atom-radiation interaction

We believe the result we obtained in the model of the harmonic oscillator may be interesting in the case of the coupling of the atom with the electromagnetic field. Since the argument of Section 3.2 could be transposed to the atom-radiation case, it seems likely to us that the result we obtained for the Hamiltonian (1) and the particular function g should extend, not only to more general g but to the photon-atom Hamiltonian.

In this more physical context, we will now indicate where a singularity of g may come from, whether it has a physical origin and where it could be located in the energy complex plane.

An example of a matrix element of the Hamiltonian and its poles

Let us now consider an atom. Let $|f\rangle$ denote its fundamental state with energy E_f, $|e\rangle$ an excited state with energy E_e, and consider the emission of a photon with energy E and quantum numbers j,m.

$$e \to f \ + \ |E,j,m\rangle.$$

As a model for studying that transition, one may use the Hamiltonian

$$(E_f|f\rangle\langle f| + E_e|e\rangle\langle e|)\otimes 1 + 1\otimes H_{\rm rad} + |f\rangle\langle e|\otimes c^*(g) + |e\rangle\langle f|\otimes c(g),$$

with $g(k) := \Big(\langle f|\otimes\langle|k|,j,m|\Big)H\Big(|e\rangle\otimes|\Omega_{\rm rad}\rangle\Big)$. Such matrix elements are given for example in [9]. For the hydrogen atom, e being the level $n=2$ with $j=1$, $m=0$, $g(E)$ is shown to be proportional to

$$\frac{E}{[(1+(2a\hbar^{-1}c^{-1}E/3)^2]^2}, \qquad (16)$$

where a is the Bohr radius. Here the pole of g in $\Im z \le 0$ is purely imaginary and the order of magnitude of its modulus is 5 keV. It is far greater than the width of the atom states. That is why the poles of the matrix elements of the resolvent which it would generate could be ignored. But although the effect is small, taking it into account might be of interest from the theoretical point of view.

Position of the poles of g and atomic distances

The connection between the distance to the real axis of the poles of g and the atomic distances may be seen more generally as follows. Let ψ_0 be the wavefunction of the fundamental state and ψ_1 that of a given excited state. Assuming the photon to be scalar, the matrix element $g(k) := \langle 0,k|H|1\rangle$ is proportional to the Fourier transform of the function $\overline{\psi}_0(r)\partial_r\psi_1(r)$, and thus the imaginary parts of the poles are connected with the spatial extension of the wavefunctions. Hence they have a physical meaning.

3.4.2 A possible application in nuclear physics

The situation is different if the quantity a in formula (16), or the spatial extension of the states, is no more of the order of the Angstrom but rather of that of the Fermi. A mechanism similar to the one we exposed could then produce poles of the resolvent matrix elements the imaginary parts of which would now correspond to energies of the order of 1 F^{-1} instead of 1 $\mathrm{\AA}^{-1}$. They are now of the same order of magnitude as the width of certain resonances. As an example, the resonance γp at 1500 MeV has a width of the order of 100 MeV. In such a situation, the correspondence resonance-pole would be altered in a significant way.

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